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Volume comparison for Lorentzian warped products with integral curvature bounds

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Abstract

In this paper, we establish a generalized volume comparison theorem for Lorentzian warped products, where we have an integral curvature bound condition.

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Keywords: Volume comparison; Riccati equation

1. Introduction

The volume comparison theorem for metric ball $B(x, r)$ has been one of the most important tools in the comparison geometry of Riemannian manifolds, and it has been developed in various directions [3,11,12,14].

Most such developments have their basis in the Bishop–Gromov volume comparison theorem [10] and the main tools for these investigations have been Jacobi fields and index form arguments, even though the Riccati equation techniques are recently used very often for the various volume calculations.

Inspired by these works in Riemannian geometry, Ehrlich et al. [7] employed the Riccati equation techniques to compare the volume of the compact sets in a Lorentzian manifold with that of the corresponding compact region in a Lorentzian space form. The Riccati equation which was used in [7] has been called the 'Raychaudhuri equation' in General Relativity and it plays a crucial role in the proofs of the singularity theorems of spacetime.

One of the main assumptions in the spacetime version of the Bishop–Gromov volume comparison given in [7] is on the Ricci curvature bounds as follows:

$$\mathrm{Ric}(v, v) \geq (n-1)k > 0,$$

for all timelike unit vectors v and for some $k > 0$.

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This curvature condition implies the 'strong energy condition' which appears in many important results in the classical General Relativity.

But, recent observations in cosmology indicate that two thirds of the density in the Universe is contained in a new form which violates the strong energy condition and is referred to as 'dark energy' (see [6]). In addition, the strong energy condition implies that there can be no acceleration in the expanding Universe, which is also violated by the contemporary observations in cosmology (see [13,15]).

For these reasons, the strong energy condition is not used very often any longer. Indeed, if we go into the quantum world, it is possible that some other energy conditions are also violated locally.

Alternatively, however, the averaged energy density of matter in the Universe turns out to be meaningful in contemporary observational cosmology (see [4]). From these points of view, it is interesting to introduce the curvature invariant in [12], which is defined to measure the amount of curvature below a given number in an integral sense.

In [11,12], the authors generalized the classical Bishop–Gromov volume comparison estimate to a situation where one only has an integral bound for the part of the Ricci curvature which lies below a given number. Inspired by this work, we shall establish the spacetime version of this generalized volume comparison theorem by following the arguments in [11].

In addition to the curvature conditions, we also consider a more generalized model space for comparison which need not have constant curvature throughout the spacetime. This generalization is reasonable because the physical Universe need not have constant curvature throughout the *spacetime*. Indeed, the spacetime with constant curvature represents a universe which contains nothing but the vacuum energy (see p. 328 in [5] for details).

But, in our Universe, we have ordinary matter and radiation as well as a possible vacuum energy. To describe the real Universe, it turns out to be more reasonable to posit that the Universe is *spatially* homogeneous and isotropic but evolving in time. We therefore consider the generalized Robertson–Walker spacetime with warped product metric as our model space for comparison.

Indeed, the spacetime with warped product metric is one of the typical examples of globally hyperbolic spacetime and has played an important role in the asymptotic behavior of spacetimes [1].

Consequently, it is the purpose of this paper that we establish a generalized volume comparison theorem for Lorentzian warped products, where we have an integral curvature bound condition.

2. Preliminaries and main theorem

Let (M, g) be an arbitrary time-oriented spacetime of dimension $n \geq 2$ with a nondegenerate metric g of signature $(-, +, \dots, +)$ for each tangent space of M.

We say that $x \in M$ *chronologically (resp. causally) precedes* $y \in M$ if there is a timelike (resp. causal) path from x to y. We can then define the *chronological future* $I^+(x)$ and the *causal future* $J^+(x)$ as follows.

$$I^+(x) = \{y \in M | x \text{ chronologically precedes } y\},$$
$$J^+(x) = \{y \in M | x \text{ causally precedes } y\}.$$

Recall that the Lorentzian distance function $d_x : M \to [0, +\infty]$ is defined by setting

$$d_x(y) = \sup\{L(c) | c \text{ is a future directed causal curve from } x \text{ to } y\},$$

if $y \in J^+(x)$ and setting $d_x(y) = 0$ if y is not in $J^+(x)$. Here, $L(c)$ is the length of the causal curve $c : [a, b] \to M$ defined by

$$L(c) = \int_a^b \sqrt{-g(c'(t), c'(t))}\mathrm{d}t.$$

We say that a spacetime is *globally hyperbolic* if it admits a Cauchy surface which is, by definition, a subset of M satisfying that every inextendible causal curve intersects it exactly once (by definition, a piecewise smooth curve $c : [0, b) \to M$ is 'extendible' provided that it has a continuous extension $\tilde{c} : [0, b] \to M$ for $b \leq \infty$. Of course, c is called 'inextendible' if it is not extendible).

It is indicated in [2] that Geroch [9] has established the following important structure theorem for globally hyperbolic spacetimes.

Theorem 2.1 (*[2]*). *If (M, g) is a globally hyperbolic spacetime of dimension n, then M is homeomorphic to $\mathbb{R} \times S$, where S is an $(n-1)$-dimensional topological submanifold of M and for each t, $\{t\} \times S$ is a Cauchy surface.*

Unlike the case of complete Riemannian manifolds, we are not able to connect a chronologically related pair of points by a maximal timelike geodesic segment only with the assumption of timelike geodesic completeness. Globally hyperbolic spacetimes, however, have the desired property that for any given pair of points x, y with $y \in J^+(x)$, there exists a maximal causal geodesic segment from x to y. This property is, of course, very important in order to do many standard geometric constructions and we, therefore, always assume that (M, g) is globally hyperbolic.

Now we introduce some notations as in [7,8].

First, for each $x \in M$, let $\mathrm{Fut}(T_x M)$ denote the set of all future directed timelike vectors $v \in T_x M$ such that $\exp_x v$ is defined. Then for any unit vector $v \in \mathrm{Fut}(T_x M)$, we have a timelike radial geodesic $\gamma_v(t) = \exp_x tv$ with $\gamma_v(0) = x$, $\gamma_v'(0) = v$. Put

$$\mathrm{cut}_v(x) = \sup\{t \geq 0 | L(\gamma_v|_{[0,t]}) = d_x(\gamma_v(t))\} > 0,$$

where d_x is the Lorentzian distance function which is continuous and finite valued, since (M, g) is globally hyperbolic.

In order to obtain finite integrals over distance wedges, we let, as in [7], $S(x) = \{v \in \mathrm{Fut}(T_x M) | g(v, v) = -1\}$ and for any compact set D of $S(x)$, define the *D-distance wedge* $B_x^D(r)$ as follows:

$$B_x^D(r) = \{\exp_x tv | v \in D, 0 \leq t \leq r\}.$$

Note also that if we denote $\mathrm{inj}_D(x) = \inf\{\mathrm{cut}_v(x) | v \in D\}$, then there exists $w \in D$ with $\mathrm{cut}_w(x) = \mathrm{inj}_D(x) > 0$, since $v \to \mathrm{cut}_v(x)$ is lower semicontinuous.

Now we can establish our warped product comparison model as follows:

Let first h be the Riemannian metric induced on $S(x)$ as a subset of $T_x M$ with the Minkowski metric induced by $g|_{T_x M}$.

Consider then a warped product space $\overline{M} = (0, a) \times_f S(x)$ with Lorentzian metric $\overline{g} = -\mathrm{d}t^2 + (f(t))^2 h$, where $0 < a < \mathrm{inj}_D(x)$ and f is a warping function satisfying the Jacobi differential equation

$$f''(t) + K(t)f(t) = 0, \qquad f(0) = 0, \qquad f'(0) = 1.$$

In the above, we set $f(t) > 0$ on $(0, a)$ and $K : (0, a) \to \mathbb{R}$ is a continuous function which serves as a prescribed curvature function. Indeed, it is well known that for any unit timelike vector $\overline{u}$ at $(t, \overline{x}) \in \overline{M}$ which is normal to $\{t\} \times S(x)$, we have the Ricci curvature of $\overline{M}$:

$$\mathrm{Ric}(\overline{u}, \overline{u}) = (n-1)K(t)$$

(see [8]). The D-distance wedge $\overline{B}^D(r)$ in $\overline{M}$ which corresponds to $B_x^D(r)$ in M is defined as follows.

$$\overline{B}^D(r) = \{(t, v) \in \overline{M} (= (0, a) \times_f S(x)) | 0 \leq t \leq r, v \in D\}.$$

Now, by the arguments in Section 4 in [7] and [8], we can express the volume $V_x^D(r)$(resp. $\overline{V}^D(r)$) of $B_x^D(r)$(resp. $\overline{B}^D(r)$) as follows.

$$V_x^D(r) = \int_0^r \int_D \omega(t, v) \mathrm{d}v \mathrm{d}t,$$
$$\overline{V}^D(r) = \int_0^r \int_D \omega_K(t) \mathrm{d}v \mathrm{d}t,$$

where $\mathrm{d}v$ is the volume element of $S(x)$ with the induced metric h in $T_x M$.

In the above, we know that $\omega(t, v) = \{j_v(t)\}^{n-1}$ for some positive function $j_v(t)$ which is defined by a Jacobi tensor and satisfies

$$j_v(t) > 0 \quad \text{on } (0, \mathrm{inj}_D(x)) \qquad \text{and} \qquad j_v(0) = 0, \quad j_v'(0) = 1$$

(for details, see [7]).

We also know that $\omega_K(t) = \{f(t)\}^{n-1}$, where we recall that f satisfies the properties that $f(t) > 0$ on $(0, a)$ and $f(0) = 0$, $f'(0) = 1$ (see [8]).

Furthermore, if we let

$$h_v(t) = (n-1)\frac{j_v'(t)}{j_v(t)} = \frac{\omega'(t, v)}{\omega(t, v)},$$

then we obtain the following inequality from the Raychaudhuri equation of General Relativity (see [7]).

$$h_v'(t) + \frac{h_v^2(t)}{n-1} \leq -\mathrm{Ric}_-(t, v), \tag{2.1}$$

where $\mathrm{Ric}_-(t, v) = \inf\{\mathrm{Ric}(w, w) | w$ is a timelike unit vector in $T_y M$, $y = \exp_x tv\}$. Moreover, it is easy to check that if we let

$$h_K(t) = (n-1)\frac{f'(t)}{f(t)} = \frac{\omega_K'(t)}{\omega_K(t)},$$

then $h_K(t)$ satisfies the following equation.

$$h_K'(t) + \frac{h_K^2(t)}{n-1} = -(n-1)K(t). \tag{2.2}$$

Note that both $h_v(t)$ and $h_K(t)$ have a pole of order one at $t = 0$ and $\lim_{t\to 0}(h_K(t) - h_v(t)) = 0$ for each v.

In order to estimate the volume comparison, we need to define the curvature invariant which measures the averaged quantities of the Ricci curvature below $(n-1)K(t)$ as follows;

First, we let

$$\rho(y)(= \rho(t, v)) = \max\{(n-1)K(t) - \mathrm{Ric}_-(t, v), 0\},$$

where $y = \exp_x(tv)$, $v \in S(x)$ for some $t \geq 0$.

Then we define for any given $R < \mathrm{inj}_D(x)$ and $p > 0$,

$$k(x, p, D, R, K) = \int_0^R \int_D (\rho(t, v) + K_-(t))^p \omega \mathrm{d}v\mathrm{d}t,$$

where $K_-(t) = \max\{-K(t), 0\}$.

Now we are in a position to state our main result as follows.

Theorem 2.2. *Let (M, g) be a globally hyperbolic spacetime of dimension $n \geq 2$ and $(\overline{M}(= (0, a) \times_f S(x)), \overline{g})$ be a warped product comparison model as above. Then given $p > n/2$, and $R < a < \mathrm{inj}_D(x)$, there exists a constant $C(n, p, K, R, D)$ which is nondecreasing in R such that for $r < R$ we have*

$$\left(\frac{V_x^D(R)}{\overline{V}^D(R)}\right)^{\frac{1}{2p}} - \left(\frac{V_x^D(r)}{\overline{V}^D(r)}\right)^{\frac{1}{2p}} \leq C(n, p, K, R, D)k(x, p, D, R, K)^{\frac{1}{2p}},$$

where $V_x^D(r)$(resp. $\overline{V}^D(r)$) is the volume of the D-distance wedge $B_x^D(r)$(resp. $\overline{B}^D(r)$).

Furthermore, when $r = 0$, we obtain

$$\overline{V}_x^D(R) \leq (1 + C(n, p, K, R, D)k(x, p, D, R, K)^{\frac{1}{2p}})^{2p}\overline{V}^D(R).$$

Remark 2.1. (1) When we have $\mathrm{Ric}(v, v) \geq (n-1)K(t) \geq 0$ on $(0, a)$ for all timelike vectors v, we obtain $k(x, p, D, R, K) = 0$ in the above theorem and the inequalities in Theorems 4.3 and 4.4 in [7] follow immediately.

(2) If we have $K(t) \leq 0$ on $(0, a)$, then we will see later (Remark 3.1) that the above theorem remains true when we replace the curvature invariant $k(x, p, D, R, K)$ by

$$\|\rho\|^p_{L^p(B_x^D(R))} \left(= \int_0^R \int_D \rho(t, v)^p \omega \mathrm{d}v\mathrm{d}t\right).$$

3. Proof of the main theorem

Our proof basically follows along the lines in [11].

We first define $\psi(t,v)=\max\{h_v(t)-h_K(t),0\}$ for any $t\in[0,a)$, $v\in D$ and put $\rho_K(t,v)=\rho(t,v)+K_-(t)$. Then from (2.1) and (2.2), we know that ψ satisfies

$$\psi'+\frac{\psi^2}{n-1}+\frac{2h_K\psi}{n-1}\leq\rho. \tag{3.1}$$

We also know that $\psi(0,v)=0$ for any v. By modifying the proof of Theorem 2.1 in [11], we now show the following

Lemma 3.1. *For any $r<a$, $p>n/2$ and fixed $v\in D$, we have*

$$\int_0^r\psi^{2p}(t,v)\omega\mathrm{d}t\leq C_K(n,p,r)\int_0^r\rho_K^p(t,v)\omega\mathrm{d}t,$$

where $C_K(n,p,r)$ is a constant depending on the function K, n, p, and r.

Proof. As in [11], we multiply (3.1) by $\psi^{2p-2}\omega$ and integrate from 0 to r, then we get

$$\frac{1}{2p-1}\psi^{2p-1}(r)+\left(\frac{1}{n-1}-\frac{1}{2p-1}\right)\int_0^r\psi^{2p}\omega+\left(\frac{2}{n-1}-\frac{1}{2p-1}\right)\int_0^r\psi^{2p-1}h_K\omega\leq\int_0^r\rho\psi^{2p-2}\omega.$$

Since $h_K(t)(=\frac{f'}{f})$ is continuous on $(0,r]$ and $h_K(0)=+\infty$, we can put $h_K^{\min}(r):=\min_{t\in[0,r]}\{h_K(t)\}\in\mathbb{R}$.

Thus, the above inequality implies

$$\left(\frac{1}{n-1}-\frac{1}{2p-1}\right)\int_0^r\psi^{2p}\omega\leq\int_0^r\rho\psi^{2p-2}\omega-h_K^{\min}(r)\left(\frac{2}{n-1}-\frac{1}{2p-1}\right)\int_0^r\psi^{2p-1}\omega. \tag{3.2}$$

Now we consider two cases as follows.

Case 1: $h_K^{\min}(r)\geq 0$.

In this case, from (3.2) and by using the Hölder inequality, we have

$$\begin{aligned}\left(\frac{1}{n-1}-\frac{1}{2p-1}\right)\int_0^r\psi^{2p}\omega&\leq\int_0^r\rho\psi^{2p-2}\omega\\&\leq\left(\int_0^r\rho^p\omega\right)^{\frac{1}{p}}\left(\int_0^r\psi^{2p}\omega\right)^{1-\frac{1}{p}},\end{aligned}$$

from which we immediately obtain the desired result with ρ instead of $\rho_K(\geq\rho)$:

$$\int_0^r\psi^{2p}\omega\leq\left(\frac{1}{n-1}-\frac{1}{2p-1}\right)^{-p}\int_0^r\rho^p\omega.$$

Case 2: $h_K^{\min}(r)<0$.

Let

$$H^+:=\{t\in(0,r)\mid h_K(t)\geq 0\},$$
$$H^-:=\{t\in(0,r)\mid h_K(t)<0\}.$$

We now estimate $(\psi^{2p-1}\omega)(t)$ for any $t\in(0,r)=H^+\bigcup H^-$.

First, we consider the case $t\in H^+$.

Put $b=\sup\{s|s<t,h_K(s)<0\}$. (If $\{s|s<t,h_K(s)<0\}$ is empty, then we just put $b=0$.) Then it is easy to check that $h_K(s)>0$ on (b,t) and that $h_K(b)=0$.

We also introduce two auxiliary functions $\tilde{h}_K:=(h_K)_+$ and $\tilde{\psi}:=(h_v-\tilde{h}_K)_+$, where $u_+=\max\{u,0\}$ for any function u. Since $\tilde{\psi}(s)=(h_v)_+(s)$ if $s\in H^-$ and $(h_v)_+$ satisfies

$$(h_v)'_++\frac{(h_v)_+^2}{n-1}\leq(-\mathrm{Ric}_-)_+,$$

we can easily check that

$$\tilde{\psi}' + \frac{\tilde{\psi}^2}{n-1} + \frac{2\tilde{h}_K\tilde{\psi}}{n-1} \le \rho_K, \qquad (3.3)$$
$$\tilde{\psi}(0) = 0.$$

Multiplying this inequality by $\tilde{\psi}^{2p-2}\omega$ and integrating from 0 to b, we obtain

$$\frac{1}{2p-1}(\tilde{\psi}^{2p-1}\omega)(b) + \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)\int_0^b \tilde{\psi}^{2p}\omega + \left(\frac{2}{n-1} - \frac{1}{2p-1}\right)\int_0^b \tilde{\psi}^{2p-1}\tilde{h}_K\omega \le \int_0^b \rho_K\tilde{\psi}^{2p-2}\omega. \qquad (3.4)$$

Note that $\tilde{\psi}(b) = \psi(b)$ since $h_K(b) = 0$ and that $\tilde{\psi} \le \psi$ since $h_K \le \tilde{h}_K$.

So from the above inequality (3.4), we have

$$\frac{1}{2p-1}(\psi^{2p-1}\omega)(b) \le \int_0^b \rho_K\tilde{\psi}^{2p-2}\omega \le \left(\int_0^b \rho_K^p\omega\right)^{\frac{1}{p}}\left(\int_0^b \psi^{2p}\omega\right)^{1-\frac{1}{p}}. \qquad (3.5)$$

Now multiplying the inequality (3.1) by $\psi^{2p-2}\omega$ and integrating from b to t, we obtain

$$\frac{1}{2p-1}(\psi^{2p-1}\omega)(t) - \frac{1}{2p-1}(\psi^{2p-1}\omega)(b) + \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)\int_b^t \psi^{2p}\omega + \left(\frac{2}{n-1} - \frac{1}{2p-1}\right)\int_b^t \psi^{2p-1}h_K\omega \le \int_b^t \rho_K\psi^{2p-2}\omega. \qquad (3.6)$$

Thus we have

$$\frac{1}{2p-1}(\psi^{2p-1}\omega)(t) \le \frac{1}{2p-1}(\psi^{2p-1}\omega)(b) + \int_b^t \rho_K\psi^{2p-2}\omega \le 2\left(\int_0^r \rho_K^p\omega\right)^{\frac{1}{p}}\left(\int_0^r \psi^{2p}\omega\right)^{1-\frac{1}{p}} \quad \text{(by (3.5))}. \qquad (3.7)$$

We next consider the case $t \in H^-$.

Now, we let $c = \sup\{s | s < t, h_K(s) > 0\}$ and note that $h_K(c) = 0$. Then by the same arguments as in (3.4) and (3.5), we obtain

$$\frac{1}{2p-1}(\psi^{2p-1}\omega)(c) \le \left(\int_0^r \rho_K^p\omega\right)^{\frac{1}{p}}\left(\int_0^r \psi^{2p}\omega\right)^{1-\frac{1}{p}}. \qquad (3.8)$$

Now note that if we drop the ψ^2 term in the inequality (3.1) and multiply through by ψ^{2p-2}, then we have

$$\psi'\psi^{2p-2} + \frac{2}{n-1}\psi^{2p-1}h_K^{\min}(r) \le \rho\psi^{2p-2}.$$

We multiply this by $(2p-1)$ and the integrating factor $\phi(t) = \exp(h_K^{\min}(r)\frac{2(2p-1)}{n-1}t) < 1$ and write this as

$$(\phi\psi^{2p-1})' \le (2p-1)\phi\rho\psi^{2p-2} \le (2p-1)\rho\psi^{2p-2}.$$

If we multiply this inequality by ω and integrate from c to t, we get

$$(\phi\psi^{2p-1}\omega)\Big|_c^t - \int_c^t h\phi\psi^{2p-1}\omega \le (2p-1)\int_c^t \rho\psi^{2p-2}\omega, \qquad (3.9)$$

which can be reduced to

$$(\phi\psi^{2p-1}\omega)\Big|_c^t \leq (2p-1)\left(\int_c^t h_+\psi^{2p-1}\omega + \int_c^t \rho\psi^{2p-2}\omega\right).$$

Here, we used the fact that $\phi(t) < 1, h < h_+$.

Note that if we follow the same process as in Case 1 with (3.3) instead of (3.1), then we have

$$\int_0^r \tilde{\psi}^{2p}\omega \leq \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)^{-p} \int_0^r \rho_K^p\omega.$$

So, noting that $h_+ = \tilde{\psi}$ on (c, t), we have the following inequality.

$$\begin{aligned}\int_c^t h_+\psi^{2p-1}\omega &\leq \left(\int_0^r \tilde{\psi}^{2p}\omega\right)^{\frac{1}{2p}} \left(\int_0^r \psi^{2p}\omega\right)^{1-\frac{1}{2p}} \\ &\leq \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)^{-\frac{1}{2}} \left(\int_0^r \rho_K^p\omega\right)^{\frac{1}{2p}} \left(\int_0^r \psi^{2p}\omega\right)^{1-\frac{1}{2p}}. \end{aligned} \tag{3.10}$$

Consequently, we obtain from (3.9) that

$$\begin{aligned}(\phi\psi^{2p-1}\omega)(t) &\leq (\psi^{2p-1}\omega)(c) + (2p-1)\left(\int_c^t h_+\psi^{2p-1}\omega + \int_c^t \rho\psi^{2p-2}\omega\right) \\ &\leq (2p-1)\left(\int_0^r \rho_K^p\omega\right)^{\frac{1}{p}} \left(\int_0^r \psi^{2p}\omega\right)^{1-\frac{1}{p}} \quad \text{(by (3.8))} \\ &\quad + (2p-1)\left(\frac{1}{n-1} - \frac{1}{2p-1}\right)^{-\frac{1}{2}} \left(\int_0^r \rho_K^p\omega\right)^{\frac{1}{2p}} \left(\int_0^r \psi^{2p}\omega\right)^{1-\frac{1}{2p}} \quad \text{(by (3.10))} \\ &\quad + (2p-1)\left(\int_0^r \rho_K^p\omega\right)^{\frac{1}{p}} \left(\int_0^r \psi^{2p}\omega\right)^{1-\frac{1}{p}}. \end{aligned}$$

The above inequality together with (3.7) gives the estimate of $(\psi^{2p-1}\omega)(t)$ for any $t \in (0, r)$ as follows.

$$(\psi^{2p-1}\omega)(t) \leq C_1(p, n, r, K)\left\{\left(\int_0^r \rho_K^p\omega\right)^{\frac{1}{p}} \left(\int_0^r \psi^{2p}\omega\right)^{1-\frac{1}{p}} + \left(\int_0^r \rho_K^p\omega\right)^{\frac{1}{2p}} \left(\int_0^r \psi^{2p}\omega\right)^{1-\frac{1}{2p}}\right\}.$$

Now we have arrived at the same situation as in [11] and we can apply the rest of the arguments in [11] (see pp. 280–281 in [11]) to conclude the desired result. □

Remark 3.1. If we have $K(t) \leq 0$, then $f''(t) = -K(t)f(t) \geq 0$. Thus we know that $f'(t)$ is increasing and is positive, since $f'(0) = 1$. This, in turn, implies that $h_K(t) = (n-1)\frac{f'(t)}{f(t)}$ is positive. So we have $h_K^{\min}(r) \geq 0$ and obtain the result in Lemma 3.1 with ρ instead of ρ_K as mentioned in Remark 2.1(2). Note also that if we have $K(t) \geq 0$, then clearly we have $\rho = \rho_K$.

Now we prove an analogue of Lemma 2.1 in [12] in order to apply Lemma 3.1 to our settings.

Lemma 3.2. *For the volume ratio* $\frac{V_x^D(r)}{\overline{V}^D(r)}$, $(0 < r < a)$ *we have*

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{V_x^D(r)}{\overline{V}^D(r)}\right) \leq C_K(n, r)\left(\frac{V_x^D(r)}{\overline{V}^D(r)}\right)^{1-\frac{1}{2p}} \left(\int_{B_x^D(r)} \psi^{2p}\mathrm{dvol}\right)^{\frac{1}{2p}} \{\overline{V}^D(r)\}^{-\frac{1}{2p}}.$$

Proof. We first calculate the following, as in [12].

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\frac{\int_D \omega(r, v)\mathrm{d}v}{\int_D \omega_K(r)\mathrm{d}v} &\leq \frac{1}{\mathrm{vol}_h(D)}\int_D \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\omega(r, v)}{\omega_K(r)}\right)\mathrm{d}v \\ &\leq \frac{1}{\mathrm{vol}_h(D)}\int_D \psi\frac{\omega}{\omega_K}\mathrm{d}v. \end{aligned}$$

Thus for $t \leq r$, we have

$$\frac{\int_{D(x)} \omega(r, v)\mathrm{d}v}{\int_D \omega_K(r)\mathrm{d}v} - \frac{\int_D \omega(t, v)\mathrm{d}v}{\int_D \omega_K(t)\mathrm{d}v} \leq \frac{1}{\mathrm{vol}_h(D)} \int_t^r \int_D \psi \frac{\omega}{\omega_K} \mathrm{d}v\mathrm{d}s,$$

which implies

$$\begin{aligned}
&\int_D \omega(r, v)\mathrm{d}v \int_D \omega_K(t)\mathrm{d}v - \int_D \omega_K(r)\mathrm{d}v \int_D \omega(t, v)\mathrm{d}v \\
&\quad \leq \frac{1}{\mathrm{vol}_h(D)} \left(\int_D \omega_K(t)\mathrm{d}v\right) \left(\int_D \omega_K(r)\mathrm{d}v\right) \int_t^r \int_D \psi \frac{\omega}{\omega_K} \mathrm{d}v\mathrm{d}s \\
&\quad \leq \left(\int_D \omega_K(r)\mathrm{d}v\right) \int_t^r \int_D \psi \frac{\omega_K(t)}{\omega_K(s)} \omega(s, v)\mathrm{d}v\mathrm{d}s \\
&\quad = \mathrm{vol}_h(D) \int_t^r \int_D \frac{\omega_K(t)\omega_K(r)}{\omega_K(s)} \psi \omega(s, v)\mathrm{d}v\mathrm{d}s.
\end{aligned} \tag{3.11}$$

Now we consider $\min_{t \leq s \leq r}\{\omega_K(s)\}$ and note that $\min_{t \leq s \leq r}\{\omega_K(s)\} = \omega_K(t)$ for sufficiently small $r > 0$, since $\omega_K(0) = 0$ and $\omega_K(t)$ is increasing for small $t > 0$. (This follows immediately from $\omega'(t) = (n-1) f^{n-2}(t) f'(t) > 0$ for small $t > 0$.)

Thus we can put

$$F_r(t) = \frac{\omega_K(t)\omega_K(r)}{\min\limits_{t \leq s \leq r}\{\omega_K(s)\}},$$

which is well defined (for sufficiently small r, we know that $F_r(t) = \omega_K(r)$).

So we have, from (3.11),

$$\begin{aligned}
&\int_D \omega(r, v)\mathrm{d}v \int_D \omega_K(t)\mathrm{d}v - \int_D \omega_K(r)\mathrm{d}v \int_D \omega(t, v)\mathrm{d}v \\
&\quad \leq \mathrm{vol}_h(D) F_r(t) \int_t^r \int_D \psi \omega \mathrm{d}v\mathrm{d}s \\
&\quad \leq \mathrm{vol}_h(D) F_r(t) \left(\int_t^r \int_D \psi^{2p} \omega \mathrm{d}v\mathrm{d}s\right)^{\frac{1}{2p}} \left(\int_t^r \int_D \omega \mathrm{d}v\mathrm{d}s\right)^{1-\frac{1}{2p}} \\
&\quad \leq \mathrm{vol}_h(D) F_r(t) \left(\int_{B_x^D(r)} \psi^{2p} \mathrm{dvol}\right)^{\frac{1}{2p}} (\mathrm{vol}(B_x^D(r)))^{1-\frac{1}{2p}}.
\end{aligned}$$

Now we calculate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{V_x^D(r)}{\overline{V}^D(r)}\right) = \frac{1}{\{\overline{V}^D(r)\}^2} &\left\{\left(\int_D \omega(r, v)\mathrm{d}v\right)\left(\int_0^r \int_D \omega_K(t)\mathrm{d}v\mathrm{d}t\right)\right. \\
&\left. - \left(\int_D \omega_K(r)\mathrm{d}v\right)\left(\int_0^r \int_D \omega(t, v)\mathrm{d}v\mathrm{d}t\right)\right\}.
\end{aligned}$$

Observe that the numerator can be written as

$$\begin{aligned}
&\int_0^r \left\{\left(\int_D \omega(r, v)\mathrm{d}v\right)\left(\int_D \omega_K(t)\mathrm{d}v\right) - \left(\int_D \omega_K(r)\mathrm{d}v\right)\left(\int_D \omega(t, v)\mathrm{d}v\right)\right\} \mathrm{d}t \\
&\quad \leq \int_0^r \left\{\mathrm{vol}_h(D) F_r(t) \left(\int_{B_x^D(r)} \psi^{2p} \mathrm{dvol}\right)^{\frac{1}{2p}} (\mathrm{vol}(B_x^D(r)))^{1-\frac{1}{2p}}\right\} \mathrm{d}t \\
&\quad = \mathrm{vol}_h(D) (V_x^D(r))^{1-\frac{1}{2p}} \left(\int_{B_x^D(r)} \psi^{2p} \mathrm{dvol}\right)^{\frac{1}{2p}} \int_0^r F_r(t)\mathrm{d}t.
\end{aligned}$$

Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{V_x^D(r)}{\overline{V}^D(r)}\right) \le \frac{\mathrm{vol}_h(D)(V_x^D(r))^{1-\frac{1}{2p}}(\int_{B_x^D(r)}\psi^{2p}\mathrm{dvol})^{\frac{1}{2p}}\int_0^r F_r(t)\mathrm{d}t}{\{\overline{V}^D(r)\}^2}$$
$$= \left(\frac{V_x^D(r)}{\overline{V}^D(r)}\right)^{1-\frac{1}{2p}}\left(\int_{B_x^D(r)}\psi^{2p}\mathrm{dvol}\right)^{\frac{1}{2p}}\frac{\mathrm{vol}_h(D)\int_0^r F_r(t)\mathrm{d}t}{\overline{V}^D(r)}\{\overline{V}^D(r)\}^{-\frac{1}{2p}}. \tag{3.12}$$

Now we observe that

$$\frac{\mathrm{vol}_h(D)\int_0^r F_r(t)\mathrm{d}t}{\overline{V}^D(r)} = \frac{\mathrm{vol}_h(D)\int_0^r F_r(t)\mathrm{d}t}{\int_D\int_0^r \omega_K(s)\mathrm{d}s\mathrm{d}v} = \frac{\int_0^r F_r(t)\mathrm{d}t}{\int_0^r \omega_K(s)\mathrm{d}s}.$$

Since $F_r(s) = \omega_K(r)$ for small $r > 0$, we have

$$\frac{\int_0^r F_r(t)\mathrm{d}t}{\int_0^r \omega_K(s)\mathrm{d}s} = \frac{r\omega_K(r)}{\int_0^r \omega_K(s)\mathrm{d}s},$$

for small $r > 0$. But if $r > 0$ is small enough, then we have $\omega_K(r) \sim r^{n-1}$, which implies that

$$\frac{r\omega_K(r)}{\int_0^r \omega_K(s)\mathrm{d}s} \to n$$

as $r \to 0$. Thus, we can say that for any given r, there exists a constant $C_K(n, r) > 0$ such that

$$C_K(n, r) = \max_{0\le t\le r}\frac{\int_0^t F_t(s)\mathrm{d}s}{\int_0^t \omega_K(s)\mathrm{d}s}.$$

If we let $G(r) = \frac{V_x^D(r)}{\overline{V}^D(r)}$, then we get from (3.12) that

$$G'(r) \le G(r)^{1-\frac{1}{2p}}C_K(n, r)\{\overline{V}^D(r)\}^{-\frac{1}{2p}}\left(\int_{B_x^D(r)}\psi^{2p}\mathrm{dvol}\right)^{\frac{1}{2p}}, \tag{3.13}$$

which proves Lemma 3.2. □

Now, for any $r < R < a$ if we integrate (3.13) from r to R, we obtain (using Lemma 3.1) that

$$2pG(R)^{\frac{1}{2p}} - 2pG(r)^{\frac{1}{2p}} \le \alpha\int_r^R g(s)\mathrm{d}s,$$

where $\alpha = C_K(n, R)\{k(x, p, D, R, K)\}^{\frac{1}{2p}}$, and $g(s) = \{\overline{V}^D(s)\}^{-\frac{1}{2p}}$.

Consequently, we have

$$G(R)^{\frac{1}{2p}} - G(r)^{\frac{1}{2p}} \le \frac{C_K(n, R)}{2p}\left(\int_0^R\{\overline{V}^D(s)\}^{-\frac{1}{2p}}\mathrm{d}s\right)\{k(x, p, D, R, K)\}^{\frac{1}{2p}}.$$

Since $\overline{V}^D(s) \sim s^n$ for small s, we know that $\int_0^R\{\overline{V}^D(s)\}^{-\frac{1}{2p}}\mathrm{d}s < \infty$, if $p > n/2$.

So we can put $C(n, p, K, R, D) = \frac{C_K(n,R)}{2p}\int_0^R\{\overline{V}^D(s)\}^{-\frac{1}{2p}}\mathrm{d}s$ and get the desired estimate in Theorem 2.2. The second inequality in Theorem 2.2 follows immediately if we put $r = 0$ and use the fact $G(0) = 1$.

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